RECOGNIZING THE TOPOLOGY OF THE SPACE OF CLOSED CONVEX SUBSETS OF A BANACH SPACE

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Abstract. Let X be a Banach space and $\mathrm{Conv_H}(X)$ be the space of non-empty closed convex subsets of X, endowed with the Hausdorff metric d_H. We prove that each connected component $\mathcal{H}$ of the space $\mathrm{Conv_H}(X)$ is homeomorphic to one of the spaces: $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\mathbb{R}_+$, $Q\times\mathbb{R}_+$, l_2, or the Hilbert space $l_2(\kappa)$ of cardinality $\kappa\ge\mathfrak{c}$. More precisely, a component $\mathcal{H}$ of $\mathrm{Conv_H}(X)$ is homeomorphic to:

1. $\{0\}$ iff $\mathcal{H}$ contains the whole space X;
2. $\mathbb{R}$ iff $\mathcal{H}$ contains a half-space;
3. $\mathbb{R}\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a linear subspace of X of codimension 1;
4. $Q\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a linear subspace of X of finite codimension ≥ 2;
5. l_2 iff $\mathcal{H}$ contains a polyhedral convex subset of X but contains no linear subspace and no half-space in X;
6. $l_2(\kappa)$ for some cardinal $\kappa\ge\mathfrak{c}$ iff $\mathcal{H}$ contains no polyhedral convex subset of X.

1. Introduction

In this paper we recognize the topological structure of the space $\mathrm{Conv_H}(X)$ of non-empty closed convex subsets of a Banach space X. The space $\mathrm{Conv_H}(X)$ is endowed with the Hausdorff metric

$$\mathsf{d_H}(A,B)=\max\big\{\sup_{a\in A}\mathrm{dist}(a,B),\sup_{b\in B}\mathrm{dist}(b,A)\big\}\in[0,\infty]$$

where $\mathrm{dist}(a,B)=\inf_{b\in B}\|a-b\|$ is the distance from a point a to a subset B in X. In fact, the topology of $\mathrm{Conv_H}(X)$ can be defined directly without appealing to the Hausdorff metric: a subset $\mathcal{U}\subset\mathrm{Conv_H}(X)$ is open if and only if for every $A\in\mathcal{U}$ there is an open neighborhood U of the origin in X such that $B(A,U)\subset\mathcal{U}$ where $B(A,U)=\{A'\in\mathrm{Conv_H}(X):A'\subset A+U \text{ and } A\subset A'+U\}$. Here as usual, $A+B=\{a+b:a\in A,\ b\in B\}$ stands for the pointwise sum of sets $A,B\subset X$. In such a way, for every linear topological space X we can define the topology on the space $\mathrm{Conv_H}(X)$ of non-empty closed convex subsets of X. This topology will be called *the uniform topology* on $\mathrm{Conv_H}(X)$ because it is generated by the uniformity whose base consists of the sets

$$2^U=\{(A,A')\in\mathrm{Conv_H}(X)^2:A\subset A'+U,\ A'\subset A+U\}$$

where U runs over open symmetric neighborhoods of the origin in X.

We shall observe in Remark 4.8 that for a Banach space X the space $\mathrm{Conv_H}(X)$ is locally connected: two sets $A,B\in\mathrm{Conv_H}(X)$ lie in the same connected component of $\mathrm{Conv_H}(X)$ if and only if $\mathsf{d_H}(A,B)<+\infty$. So, in order to understand the topological structure of the hyperspace $\mathrm{Conv_H}(X)$ it suffices to recognize the topology of its connected components. This problem is quite easy if X is a 1-dimensional real space. In this case X is isometric to $\mathbb{R}$ and a connected component $\mathcal{H}$ of $\mathrm{Conv_H}(X)$ is isometric to:

1. $\{0\}$ iff $X\in\mathcal{H}$;
2. $\mathbb{R}$ iff $\mathcal{H}$ contains a closed ray;
3. $\mathbb{R}\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a bounded set.

Here $\bar{\mathbb{R}}_+=[0,+\infty)$ stands for the closed half-line.

For arbitrary Banach spaces we shall add to this list two more spaces:

4. $Q\times\bar{\mathbb{R}}_+$, where $Q=[0,1]^\omega$ is the Hilbert cube;
5. $l_2(\kappa)$, the Hilbert space having an orthonormal basis of cardinality κ.

For $\kappa=\omega$ the separable Hilbert space $l_2(\omega)$ is usually denoted by l_2. By the famous Toruńczyk Theorem [15], [16], each infinite-dimensional Banach space X of density κ is homeomorphic to the Hilbert space $l_2(\kappa)$. In particular, the Banach space l_∞ of bounded real sequences is homeomorphic to $l_2(\mathfrak{c})$. In the sequel we shall identify cardinals with the sets of ordinals of smaller cardinality and endow such sets with discrete topology. The cardinality of a set A is denoted by $|A|$.

Let X be a Banach space. As we shall see in Theorem 1, each non-locally compact connected component $\mathcal{H}$ of the space $\mathrm{Conv_H}(X)$ is homeomorphic to the Hilbert spaces $l_2(\kappa)$ of density $\kappa=\mathrm{dens}(\mathcal{H})$. This reduces the

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problem of recognizing the topology of $\mathrm{Conv}_{\mathsf{H}}(X)$ to calculating the densities of its components. In fact, separable components $\mathcal{H}$ of $\mathrm{Conv}_{\mathsf{H}}(X)$ have been characterized in [3] as components containing a polyhedral convex set.

We recall that a convex subset C of a Banach space X is *polyhedral* if C can be written as the intersection $C=\bigcap\mathcal{F}$ of a finite family $\mathcal{F}$ of closed half-spaces. A *half-space* in X is a convex set of the form $f^{-1}\big((-\infty,a]\big)$ for some real number a and some non-zero linear continuous functional $f:X\to\mathbb{R}$. The whole space X is polyhedral being the intersection $X=\bigcap\mathcal{F}$ of the empty family $\mathcal{F}=\emptyset$ of closed half-spaces.

The principal result of this paper is the following classification theorem.

Theorem 1. *Let X be a Banach space. Each connected component $\mathcal{H}$ of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ is homeomorphic to one of the spaces: $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\bar{\mathbb{R}}_+$, $Q\times\bar{\mathbb{R}}_+$, l_2, or the Hilbert space $l_2(\kappa)$ of density $\kappa\ge\mathfrak{c}$. More precisely, $\mathcal{H}$ is homeomorphic to:*

1. *$\{0\}$ iff $\mathcal{H}$ contains the whole space X;*
2. *$\mathbb{R}$ iff $\mathcal{H}$ contains a half-space;*
3. *$\mathbb{R}\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a linear subspace of X of codimension 1;*
4. *$Q\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a linear subspace of X of finite codimension ≥ 2;*
5. *l_2 iff $\mathcal{H}$ contains a polyhedral convex subset of X but contains no linear subspace and no half-space in X;*
6. *$l_2(\kappa)$ for some cardinal $\kappa\ge\mathfrak{c}$ iff $\mathcal{H}$ contains no polyhedral convex subset of X.*

Theorem 1 will be proved in Section 6 after some preliminary work done in Sections 2–5.

In Corollary 2 below we shall derive from Theorem 1 a complete topological classification of the spaces $\mathrm{Conv}_{\mathsf{H}}(X)$ for Banach spaces X with Kunen-Shelah property and $|X^*|\le\mathfrak{c}$.

A Banach space X is defined to have the *Kunen-Shelah property* if each closed convex subset $C\subset X$ can be written as intersection $C=\bigcap\mathcal{F}$ of an at most countable family $\mathcal{F}$ of closed half-spaces (in fact, this is one of seven equivalent Kunen-Shelah properties considered in [6] and [7, 8.19]). For a Banach space X with the Kunen-Shelah property we get

$$|X^*|\le|\mathrm{Conv}_{\mathsf{H}}(X)|\le|X^*|^{\omega}.$$

The upper bound $\mathrm{Conv}_{\mathsf{H}}(X)\le|X^*|^{\omega}$ follows from the definition of the Kunen-Shelah property while the lower bound $|X^*|\le|\mathrm{Conv}_{\mathsf{H}}(X)|$ follows from the observation that a functional $f\in X^*$ is uniquely determined by its polar half-space $H_f=f^{-1}\big((-\infty,1]\big)$.

It is clear that each separable Banach space has the Kunen-Shelah property. However there are also non-separable Banach spaces with that property. The first example of such Banach space was constructed by S. Shelah under $\diamondsuit_{\aleph_1}$ [13]. The second example is due to K. Kunen who used the Continuum Hypothesis to construct a non-metrizable scattered compact space K such that the Banach space $X=C(K)$ of continuous functions on K is hereditarily Lindelöf in the weak topology and thus has the Kunen-Shelah Property, see [10, p.1123]. The Kunen's space $X=C(K)$ has an additional property that its dual space $X^*=C(X)^*$ has cardinality $|X^*|=\mathfrak{c}$ (this follows from the fact that each Borel measure on the scattered compact space K has countable support). Let us remark that for every separable Banach space X the dual space X^* also has cardinality of continuum $|X^*|=\mathfrak{c}$. It should be mentioned that non-separable Banach spaces with the Kunen-Shelah property can be constructed only under certain additional set-theoretic assumptions. By [14], there are models of ZFC in which each Banach space with the Kunen-Shelah property is separable.

Corollary 1. *For a separable Banach space (more generally, a Banach space with the Kunen-Shelah property and $|X^*|=\mathfrak{c}$), each connected component $\mathcal{H}$ of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ is homeomorphic to $\{0\}$, $\mathbb{R}$, $\mathbb{R}\times\bar{\mathbb{R}}_+$, $Q\times\bar{\mathbb{R}}_+$, l_2 or l_∞. More precisely, $\mathcal{H}$ is homeomorphic to:*

1. *$\{0\}$ iff $\mathcal{H}$ contains the whole space X;*
2. *$\mathbb{R}$ iff $\mathcal{H}$ contains a half-space;*
3. *$\mathbb{R}\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a linear subspace of X of codimension 1;*
4. *$Q\times\bar{\mathbb{R}}_+$ iff $\mathcal{H}$ contains a linear subspace of X of codimension ≥ 2;*
5. *l_2 iff $\mathcal{H}$ contains a polyhedral convex set but contains no linear subspace and no half-space;*
6. *l_∞ iff $\mathcal{H}$ contains no polyhedral convex set.*

Since the space $\mathrm{Conv}_{\mathsf{H}}(X)$ is homeomorphic to the topological sum of its connected components, we can use Corollary 1 to classify topologically the spaces $\mathrm{Conv}_{\mathsf{H}}(X)$ for separable Banach spaces X (and more generally Banach spaces with the Kunen-Shelah property and $|X^*|\le\mathfrak{c}$). In the following corollary the cardinal $\mathfrak{c}$ is considered as a discrete topological space.

Corollary 2. *For a separable Banach space X (more generally, a Banach space X with the Kunen-Shelah property and $|X^*|\le\mathfrak{c}$) the space $\mathrm{Conv}_{\mathsf{H}}(X)$ is homeomorphic to the topological sum:*

1. *$\{0\}\oplus\mathbb{R}\oplus\mathbb{R}\oplus(\mathbb{R}\times\bar{\mathbb{R}}_+)$ iff $\dim(X)=1$;*
2. *$\{0\}\oplus Q\times\bar{\mathbb{R}}_+\oplus\ \mathfrak{c}\times(\mathbb{R}\oplus\mathbb{R}\times\bar{\mathbb{R}}_+\oplus l_2\oplus l_\infty)$ iff $\dim(X)=2$;*

(3) $\{0\} \oplus \mathfrak{c} \times (\mathbb{R} \oplus \mathbb{R} \times \bar{\mathbb{R}}_+ \oplus Q \times \bar{\mathbb{R}}_+ \oplus l_2 \oplus l_\infty)$ *iff* $\dim(X) \geq 3$.

Moreover, under $2^{\omega_1} > \mathfrak{c}$, *for a Banach space* X, *the space* $\mathrm{Conv}_{\mathsf{H}}(X)$ *has cardinality* $|\mathrm{Conv}_{\mathsf{H}}(X)| \leq \mathfrak{c}$ *if and only if* $|X^*| \leq \mathfrak{c}$ *and the Banach space* X *has the Kunen-Shelah property.*

Proof. The statements (1)–(3) easily follow from the classification of the components of $\mathrm{Conv}_H(X)$ given in Corollary 1 and a routine calculation of the number of components of a given topological type.

Now assume that $2^{\omega_1} > \mathfrak{c}$. If X is a Banach space with the Kunen-Shelah property and $|X^*| \leq \mathfrak{c}$, then the definition of the Kunen-Shelah property yields the upper bound

$$|\mathrm{Conv}_{\mathsf{H}}(X)| \leq |X^*|^\omega \leq \mathfrak{c}^\omega = \mathfrak{c}.$$

If $|\mathrm{Conv}_{\mathsf{H}}(X)| \leq \mathfrak{c}$, then $|X^*| \leq \mathfrak{c}$ as $|X^*| \leq |\mathrm{Conv}_{\mathsf{H}}(X)|$ (because each functional $f \in X^*$ can be uniquely identified with its polar half-space $f^{-1}\big((-\infty, 1]\big) \in \mathrm{Conv}_{\mathsf{H}}(X)$). Assuming that X fails to have the Kunen-Shelah property and applying Theorem 8.19 of [7] (see also [6]), we can find a sequence $\{x_\alpha\}_{\alpha<\omega_1} \subset X$ such that for every $\alpha < \omega_1$ the point x_α does not lie in the closed convex hull $C_{\omega_1\setminus\{\alpha\}}$ of the set $\{x_\alpha\}_{\alpha\in\omega_1\setminus\{\alpha\}}$. Now for every subset $A \subset \omega_1$ consider the closed convex hull $C_A = \overline{\mathrm{conv}}\{x_\alpha\}_{\alpha\in A}$. We claim that $C_A \neq C_B$ for any distinct subsets $A, B \subset \omega_1$. Indeed, if $A \neq B$ then the symmetric difference $(A \setminus B) \cup (B \setminus A)$ contains some ordinal α. Without loss of generality, we can assume that $\alpha \in A \setminus B$. Then $x_\alpha \in C_A \setminus C_B$ as $C_B \subset C_{\omega_1\setminus\{\alpha\}} \ni x_\alpha$. This implies that $\{C_A : A \subset \omega_1\}$ is a subset of cardinality $2^{\omega_1} > \mathfrak{c}$ in $\mathrm{Conv}_{\mathsf{H}}(X)$ and hence $|\mathrm{Conv}_{\mathsf{H}}(X)| \geq 2^{\omega_1} > \mathfrak{c}$, which is a desired contradiction. □

Among the connected components of $\mathrm{Conv}_{\mathsf{H}}(X)$ there is a special one, namely, the component $\mathcal{H}_0$ containing the singleton $\{0\}$. This component coincides with the space $\mathrm{BConv}_H(X)$ of all non-empty bounded closed convex subsets of a Banach space X. The spaces $\mathrm{BConv}_H(X)$ have been intensively studied both by topologists [9], [12] and analysts [5]. In particular, S.Nadler, J. Quinn and N.M. Stavrakas [9] proved that for a finite $n \geq 2$ the space $\mathrm{BConv}_H(\mathbb{R}^n)$ is homeomorphic to $Q \times \bar{\mathbb{R}}_+$ while K. Sakai proved in [12] that for an infinite-dimensional Banach space X the space $\mathcal{H}_0 = \mathrm{BConv}_H(X)$ is homeomorphic to a non-separable Hilbert space. Moreover, if X is separable or reflexive, then $\mathrm{dens}(\mathcal{H}_0) = 2^{\mathrm{dens}(X)}$. In the latter case the density $\mathrm{dens}^*(X^*)$ of the dual space X^* in the weak* topology is equal to the density $\mathrm{dens}(X)$ of X. Banach spaces X with $\mathrm{dens}^*(X^*) = \mathrm{dens}(X^*)$ are called DENS Banach spaces, see [7, 5.39]. By Proposition 5.40 of [7], the class of DENS Banach spaces includes all weakly Linedlöf determined spaces, and hence all weakly countably generated and all reflexive Banach spaces.

Applying Theorem 1 to describing the topology of the component $\mathcal{H}_0 = \mathrm{BConv}_H(X)$, we obtain the following classification.

Corollary 3. *The space* $\mathcal{H}_0 = \mathrm{BConv}_H(X)$ *of non-empty bounded closed convex subsets of a Banach space* X *is homeomorphic to one of the spaces:* $\{0\}$, $\mathbb{R}\times\bar{\mathbb{R}}_+$, $Q\times\bar{\mathbb{R}}_+$ *or the Hilbert space* $l_2(\kappa)$ *of density* $\kappa \geq \mathfrak{c}$. *More precisely,* $\mathrm{BConv}(X)$ *is homeomorphic to:*

(1) $\{0\}$ *iff* $\dim X = 0$;
(2) $\mathbb{R} \times \mathbb{R}^+$ *iff* $\dim X = 1$;
(3) $Q \times \bar{\mathbb{R}}_+$ *iff* $2 \leq \dim(X) < \infty$;
(4) $l_2(\kappa)$ *for some cardinal* $\kappa \in [2^{\mathrm{dens}^*(X^*)}, 2^{\mathrm{dens}(X)}]$ *iff* $\dim(X) = \infty$;
(5) $l_2(2^{\mathrm{dens}(X)})$ *if* X *is an infinite-dimensional DENS Banach space.*

Proof. This corollary will follow from Theorem 1 as soon as we check that

$$2^{\mathrm{dens}^*(X^*)} \leq \mathrm{dens}(\mathcal{H}_0) \leq |\mathcal{H}_0| \leq |\mathrm{Conv}_{\mathsf{H}}(X)| \leq 2^{\mathrm{dens}(X)}$$

for each infinite-dimensional Banach space X.

In fact, the inequality $|\mathrm{Conv}_{\mathsf{H}}(X)| \leq 2^{\mathrm{dens}(X)}$ has general-topological nature and follows from the known fact that the number of closed subsets (equal to the number of open subsets) of a topological space X does not exceed $2^{w(X)}$.

To prove that $2^{\mathrm{dens}^*(X^*)} \leq \mathrm{dens}(\mathcal{H}_0)$ we shall use a result of Plichko [11] (see also Theorem 4.12 [7]) saying that for each infinite-dimensional Banach space X there is a bounded sequence $\{(x_\alpha, f_\alpha)\}_{\alpha<\kappa} \subset X \times X^*$ of length $\kappa = \mathrm{dens}^*(X^*)$, which is biorthogonal in the sense that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) = 0$ for any distinct ordinals $\alpha, \beta < \kappa$. Let $L = \sup\{\|x_\alpha\|, \|f_\alpha\| : \alpha < \kappa\}$.

For every subset $A \subset \kappa$ consider the closed convex hull $C_A = \overline{\mathrm{conv}}(\{x_\alpha\}_{\alpha\in A})$ of the set $\{x_\alpha\}_{\alpha\in A}$. We claim that for any distinct subsets $A, B \subset \kappa$ we get $\mathsf{d}_{\mathsf{H}}(C_A, C_B) \geq \frac{1}{L}$. Indeed, since $A \neq B$ the symmetric difference $(A \setminus B) \cup (B \setminus A)$ contains some ordinal α. Without loss of generality, we can assume that $\alpha \in A \setminus B$. Then $C_B \subset f^{-1}(0)$ and hence for each $c \in C_B$ we get $\|x_\alpha - c\| \geq \frac{|f_\alpha(x_\alpha) - f_\alpha(c)|}{\|f_\alpha\|} \geq \frac{|1-0|}{L}$, which implies $\mathrm{dist}(x_\alpha, C_B) \geq \frac{1}{L}$ and hence $\mathsf{d}_{\mathsf{H}}(C_A, C_B) \geq \frac{1}{L}$ as $x_\alpha \in C_A$.

Now we see that $\mathcal{C} = \{C_A : A \subset \kappa\}$ is a closed discrete subspace in $\mathcal{H}_0$ and hence $\mathrm{dens}(\mathcal{H}_0) \geq |\mathcal{C}| = 2^\kappa = 2^{\mathrm{dens}^*(X*)}$. □

Corollaries 1 and 2 motivate the following problem.

Problem 1.1. Is $|X^*| \le \mathfrak{c}$ for each Banach space X with the Kunen-Shelah property?

Another problem concerns possible densities of the components of the space $\mathrm{Conv}_{\mathsf{H}}(X)$.

Problem 1.2. Let X be an infinite-dimensional Banach space. Is it true that each component $\mathcal{H}$ (in particular, the component $\mathcal{H}_0$) of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ has density 2^κ or $2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa\}$ for some cardinal κ?

Observe that under GCH (the Generalized Continuum Hypothesis) the answer to Problem 1.2 is trivially "yes" as under GCH all cardinals are of the form $2^{<\kappa}$ for some κ.

2. Hypermetric spaces

Because the Hausdorff distance d_{H} on $\mathrm{Conv}_{\mathsf{H}}(X)$ can take the infinite value we should work with generalized metrics called hypermetrics. The precise definition is as follows:

A *hypermetric* on a set X is a function $d : X \times X \to [0,\infty]$ satisfying the three axioms of a usual metric:

- $d(x,y) = 0$ iff $x = y$,
- $d(x,y) = d(y,x)$,
- $d(x,z) \le d(x,y) + d(y,z)$.

Here we extend the addition operation from $(-\infty,\infty)$ to $[-\infty,\infty]$ letting

$$\infty + \infty = \infty,\ -\infty + (-\infty) = -\infty,\ \infty + (-\infty) = -\infty + \infty = 0$$

and

$$x + \infty = \infty + x = \infty,\ x + (-\infty) = -\infty + x = -\infty$$

for every $x \in (-\infty,\infty)$.

A *hypermetric space* is a pair (X,d) consisting of a set X and a hypermetric d on X. It is clear that each metric is a hypermetric and hence each metric space is a hypermetric space.

In some respect, the notion of a hypermetric is more convenient than the usual notion of a metric. In particular, for any family (X_i, d_i), $i \in \mathcal{I}$, of hypermetric spaces it is trivial to define a nice hypermetric d on the topological sum $X = \oplus_{i\in\mathcal{I}} X_i$. Just let

$$d(x,y) = \begin{cases} d_i(x,y), & \text{if } x, y \in X_i; \\ \infty, & \text{otherwise.} \end{cases}$$

The obtained hypermetric space (X,d) will be called the *direct sum* of the family of hypermetric spaces (X, d_i), $i \in \mathcal{I}$.

In fact, each hypermetric space (X,d) decomposes into the direct sum of metric subspaces of X called metric components of X. More precisely, a *metric component* of X is an equivalence class of X by the equivalence relation $\sim$ defined as $x \sim y$ iff $d(x,y) < \infty$. So, the *metric component* of a point $x \in X$ coincides with the set $\mathbb{B}_{<\infty}(x) = \{x' \in X : d(x,x') < \infty\}$. The restriction of the hypermetric d to each metric component is a metric. Therefore X is the direct sum of its metric components and hence understanding the (topological) structure of a hypermetric space reduces to studying the metric (or topological) structure of its metric components.

A typical example of a hypermetric is the Hausdorff hypermetric d_{H} on the space $\mathrm{Cld}(X)$ of non-empty closed subsets of a (linear) metric space X (and the restriction of d_{H} to the subspace $\mathrm{Conv}(X) \subset \mathrm{Cld}(X)$ of non-empty closed convex subsets of X). So both $\mathrm{Cld}_{\mathsf{H}}(X) = (\mathrm{Cld}(X), \mathsf{d}_{\mathsf{H}})$ and $\mathrm{Conv}_{\mathsf{H}}(X) = (\mathrm{Conv}(X), \mathsf{d}_{\mathsf{H}})$ are hypermetric spaces.

A much simple (but still important) example of a hypermetric space is the extended real line $\overline{\mathbb{R}} = [-\infty,\infty]$ with the hypermetric

$$d(x,y) = \begin{cases} |x-y|, & \text{if } x, y \in (-\infty,\infty), \\ 0, & \text{if } x = y \in \{-\infty,\infty\}, \\ \infty, & \text{otherwise,} \end{cases}$$

which will be denoted by $|x-y|$. The hypermetric space $\overline{\mathbb{R}}$ has three metric components: $\{-\infty\}$, $\mathbb{R}$, $\{\infty\}$.

This example allows us to construct another important example of a hypermetric space. Namely, for a set Γ consider the space $\overline{\mathbb{R}}^\Gamma$ of functions from Γ to $\overline{\mathbb{R}}$ endowed with the hypermetric

$$d(f,g) = \|f-g\|_\infty = \sup_{\gamma\in\Gamma} |f(\gamma) - g(\gamma)|.$$

The obtained hypermetric space $(\overline{\mathbb{R}}^\Gamma, \|\cdot - \cdot\|_\infty)$ will be denoted by $\bar{l}_\infty(\Gamma)$. Observe that the topology of $\bar{l}_\infty(\Gamma)$ is different from the Tychonoff product topology of $\overline{R}^\Gamma$. Another reason for using the notation $\bar{l}_\infty(\Gamma)$ is that the

metric component of $\bar{l}_\infty(\Gamma)$ containing the zero function coincides with the classical Banach space $l_\infty(\Gamma)$ of bounded functions on Γ. More generally, for each $f_0 \in \bar{l}_\infty(\Gamma)$ its metric component

$$\mathbb{B}_{<\infty}(f_0) = \{f \in \bar{l}_\infty(\Gamma) : \|f - f_0\|_\infty < \infty\}$$

is isometric to the Banach space $l_\infty(\Gamma_0)$ where $\Gamma_0 = \{\gamma \in \Gamma : |f_0(\gamma)| < \infty\}$.

It turns out that for every normed space X the space $\mathrm{Conv}_{\mathsf{H}}(X)$ nicely embeds into the hypermetric space $\bar{l}_\infty(\mathbb{S}^*)$ where

$$\mathbb{S}^* = \{x^* \in X^* : \|x^*\| = 1\}$$

stands for the unit sphere of the dual Banach space X^*.

Namely, consider the function

$$\delta : \mathrm{Conv}_{\mathsf{H}}(X) \mapsto \bar{l}_\infty(\mathbb{S}^*),\ \delta : C \mapsto \delta_C$$

where $\delta_C(x^*) = \sup x^*(C)$ for $x^* \in \mathbb{S}^*$. The function δ will be called the *canonical representation* of $\mathrm{Conv}_{\mathsf{H}}(X)$.

Proposition 2.1. *For every normed space X the canonical representation $\delta : \mathrm{Conv}_{\mathsf{H}}(X) \to \bar{l}_\infty(\mathbb{S}^*)$ is an isometric embedding.*

Proof. Let $A, B \in \mathrm{Conv}_{\mathsf{H}}(X)$ be two convex sets. We should prove that $\mathsf{d}_{\mathsf{H}}(A, B) = \|\delta_A - \delta_B\|$, where

$$\|\delta_A - \delta_B\| = \sup_{x^* \in \mathbb{S}^*} |\delta_A(x^*) - \delta_B(x^*)| = \sup_{x^* \in \mathbb{S}^*} |\sup x^*(A) - \sup x^*(B)|.$$

The inequality $\|\delta_A - \delta_B\| \le \mathsf{d}_{\mathsf{H}}(A, B)$ will follow as soon as we check that $|\sup x^*(A) - \sup x^*(B)| \le \mathsf{d}_{\mathsf{H}}(A, B)$ for each functional $x^* \in \mathbb{S}^*$. This is trivial if $\mathsf{d}_{\mathsf{H}}(A, B) = \infty$. So we assume that $\mathsf{d}_{\mathsf{H}}(A, B) < \infty$. To obtain a contradiction, assume that $|\sup x^*(A) - \sup x^*(B)| > \mathsf{d}_{\mathsf{H}}(A, B)$. Then either $\sup x^*(A) - \sup x^*(B) > \mathsf{d}_{\mathsf{H}}(A, B)$ or $\sup x^*(B) - \sup x^*(A) > \mathsf{d}_{\mathsf{H}}(A, B)$. In the first case $\sup x^*(B) \ne \infty$, so we can find a point $a \in A$ with $x^*(a) - \sup x^*(B) > \mathsf{d}_{\mathsf{H}}(A, B)$. It follows from the definition of the Hausdorff metric $\mathsf{d}_{\mathsf{H}}(A, B) \ge \mathrm{dist}(a, B)$ that $\|a - b\| < x^*(a) - \sup x^*(B)$ for some point $b \in B$. Then $x^*(a) - x^*(b) \le \|x^*\| \cdot \|a - b\| < x^*(a) - \sup x^*(B)$ and hence $x^*(b) > \sup x^*(B)$, which is a contradiction.

By analogy, we can derive a contradiction from the assumption $\sup x^*(B) - \sup x^*(A) > \mathsf{d}_{\mathsf{H}}(A, B)$ and thus prove the inequality $\|\delta_A - \delta_B\| \le \mathsf{d}_{\mathsf{H}}(A, B)$.

To prove the reverse inequality $\|\delta_A - \delta_B\| \ge \mathsf{d}_{\mathsf{H}}(A, B)$ let us consider two cases:

(i) $\mathsf{d}_{\mathsf{H}}(A, B) = \infty$. To prove that $\infty = \|\delta_A - \delta_B\|$, it suffices given any number $R < \infty$ to find a linear functional $x^* \in \mathbb{S}^*$ such that $|\sup x^*(A) - \sup x^*(B)| \ge R$.

The equality $\mathsf{d}_{\mathsf{H}}(A, B) = \infty$ implies that either $\sup_{a \in A} \mathrm{dist}(a, B) = \infty$ or $\sup_{b \in B} \mathrm{dist}(b, A) = \infty$. In the first case we can find a point $a \in A$ with $\mathrm{dist}(a, B) \ge R$ and using the Hahn-Banach Theorem construct a linear functional $x^* \in \mathbb{S}^*$ that separates the convex set B from the closed R-ball $\bar{B}(a, R) = \{x \in X : \|x - a\| \le R\}$ in the sense that $\sup x^*(B) \le \inf x^*(\bar{B}(a, R))$. For this functional x^* we get $\sup x^*(A) \ge x^*(a) \ge R + \inf x^*(\bar{B}(a, R)) \ge R + \sup x^*(B)$ and thus $\sup x^*(A) - \sup x^*(B) \ge R$.

In the second case, we can repeat the preceding argument to find a linear functional $x^* \in \mathbb{S}^*$ with

$$|\sup x^*(A) - \sup x^*(B)| \ge \sup x^*(B) - \sup x^*(X) \ge R.$$

(ii) $\mathsf{d}_{\mathsf{H}}(A, B) < \infty$. To prove that $\mathsf{d}_{\mathsf{H}}(A, B) \ge \|\delta_A - \delta_B\|$ it suffices given any number $\varepsilon > 0$ to find a linear functional $x^* \in \mathbb{S}^*$ such that $|\sup x^*(A) - \sup x^*(B)| \ge \mathsf{d}_{\mathsf{H}}(A, B) - \varepsilon$. It follows from the definition of $\mathsf{d}_{\mathsf{H}}(A, B)$ that either there is a point $a \in A$ with $\mathrm{dist}(a, B) > \mathsf{d}_{\mathsf{H}}(A, B) - \varepsilon$ or else there is a point $b \in B$ with $\mathrm{dist}(b, A) > \mathsf{d}_{\mathsf{H}}(A, B) - \varepsilon$. In the first case we an use the Hahn-Banach Theorem to find a linear functional $x^* \in \mathbb{S}^*$ which separates the convex set B from the closed R-ball $\bar{B}(a, R)$ where $R = \mathsf{d}_{\mathsf{H}}(A, B) - \varepsilon$ in the sense that $\sup x^*(B) \le \inf x^*(\bar{B}(a, R)$. Then

$$\sup x^*(B) \le \inf x^*(\bar{B}(a, R)) = x^*(a) - R \le \sup x^*(A) - R$$

and hence

$$|\sup x^*(A) - \sup x^*(B)| \ge \sup x^*(A) - \sup x^*(B) \ge R = \mathsf{d}_{\mathsf{H}}(A, B) - \varepsilon.$$

The second case can be considered by analogy. □

3. Assigning cones to components of $\mathrm{Conv}_{\mathsf{H}}(X)$

In this section to each convex set C of a normed space X we assign two cones: the characteristic cone $V_C \subset X$ and the dual characteristic cone $V_C^* \subset X^*$.

We recall that a subset V of a linear space L is called a *convex cone* if $ax + by \in V$ for any points $x, y \in W$ and non-negative real numbers $a, b \in [0, +\infty)$.

For a convex subset C of a normed space X its *characteristic cone* of C is the convex cone

$$V_C = \{v \in X : \forall c \in C,\ \ c + \bar{\mathbb{R}}_+ v \subset c\}$$

lying in the normed space X, and its *dual characteristic cone* V_C^* is a closed convex cone

$$V_C^* = \{x^* \in X^* : \sup x^*(C) < \infty\}$$

which lies in the dual Banach space X^*.

It turns out that the characteristic cone V_C of a convex set C is uniquely determined by its dual characteristic cone V_C^*.

Lemma 3.1. *For any non-empty closed convex set C in a normed space X we get*

$$V_C = \bigcap_{f \in V_C^*} f^{-1}\big((-\infty, 0]\big).$$

Proof. Fix any vector $v \in V_C$ and a functional $f \in V_C^*$. Observe that for each number $t \in \bar{\mathbb{R}}_+$, we get $c + tv \in C$ and hence $f(c) + tf(v) \le \sup f(C) < \infty$, which implies that $f(v) \le 0$. This proves the inclusion $V_C \subset \bigcap_{f \in V_C^*} f^{-1}\big((-\infty, 0]\big)$.

To prove the reverse inclusion, fix any vector $v \in X \setminus V_C$. Then for some point $c \in C$ and some positive real number t we get $c + tv \notin C$. Using the Hahn-Banach Theorem, find a functional $f \in X^*$ that separates the convex set C and the point $x = c + tv$ in the sense that $\sup f(C) < f(c + tv)$. Then $f(c) \le \sup f(C) < f(c) + tf(v)$ implies that $f(v) > 0$ and $v \notin f^{-1}\big((-\infty, 0]\big)$. □

Let X be a normed space. It is easy to see that for each component $\mathcal{H}$ of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ and any two convex sets $A, B \in \mathcal{H}$ we get $C_A^* = C_B^*$. In this case Lemma 3.1 implies that $C_A = C_B$ as well. This allows us to define the *characteristic cone* $V_{\mathcal{H}}$ and the *dual characteristic cone* $V_{\mathcal{H}}^*$ of the component $\mathcal{H}$ letting $V_{\mathcal{H}} = V_C$ and $V_{\mathcal{H}}^* = V_C^*$ for any convex set $C \in \mathcal{H}$. Lemma 3.1 guarantees that

$$V_{\mathcal{H}} = \bigcap_{f \in V_{\mathcal{H}}^*} f^{-1}\big((-\infty, 0]\big),$$

so the characteristic cone $V_{\mathcal{H}}$ of $\mathcal{H}$ is uniquely determined by its dual characteristic cone $V_{\mathcal{H}}^*$.

4. The algebraical structure of $\mathrm{Conv}_{\mathsf{H}}(X)$

In this section given a normed space X we study the algebraic properties of the canonical representation $\delta : \mathrm{Conv}_{\mathsf{H}}(X) \to \bar{l}_\infty(\mathbb{S}^*)$.

Note that the space $\mathrm{Conv}_{\mathsf{H}}(X)$ has a rich algebraic structure: it possesses three interrelated algebraic operations: the multiplication by a real number, the addition, and taking the maximum. More precisely, for a real number $t \in \mathbb{R}$, and convex sets $A, B \in \mathrm{Conv}_{\mathsf{H}}(X)$ let

$t \cdot A = \{ta : a \in A\}$;
$A \oplus B = \overline{A + B}$;
$\max\{A, B\} = \overline{\mathrm{conv}}(A \cup B)$, where
$\overline{\mathrm{conv}}(Y)$ stands for the closed convex hull of a subset $Y \subset X$.

The hypermetric space $\overline{\mathbb{R}}$ also has the corresponding three operations (multiplication by a real number, addition and taking maximum) which induces the tree operations on $\bar{l}_\infty(\Gamma) = \overline{\mathbb{R}}^\Gamma$.

Proposition 4.1. *The canonical representation $\delta : \mathrm{Conv}_{\mathsf{H}}(X) \to \bar{l}_\infty(\mathbb{S}^*)$ has the following properties:*

1. $\delta(A \oplus B) = \delta(A) + \delta(B)$;
2. $\delta(\max\{A, B\}) = \max\{\delta(A), \delta(B)\}$;
3. $\delta(rA) = r\delta(A)$;

for every non-negative real number r and convex sets $A, B \in \mathrm{Conv}_{\mathsf{H}}(X)$.

Proof. The three items of the proposition follow from the three obvious equalities

$$\sup x^*(A \oplus B) = x^*(A + B) = x^*(A) + x^*(B),$$
$$\sup x^*(\overline{\mathrm{conv}}(A \cup B)) = \sup x^*(A \cup B) = \max\{\sup x^*(A), \sup x^*(B)\},$$
$$\sup x^*(rA) = r \sup x^*(A).$$

holding for every functional $x^* \in X^*$. □

Remark 4.2. Easy examples show that the last item of Proposition 4.1 does not hold for negative real numbers r. This means that the operator $\delta : \mathrm{Conv}_{\mathsf{H}}(X) \to \bar{l}_\infty(\mathbb{S}^*)$ is positively homogeneous but not homogeneous.

The operations of addition and multiplication by a real number allow us to define another important operation on $\mathrm{Conv}_{\mathsf{H}}(X)$ preserved by the canonical representation δ, namely the *Minkovski operation*

$$\mu : \mathrm{Conv}_{\mathsf{H}}(X) \times \mathrm{Conv}_{\mathsf{H}}(X) \times [0,1] \to \mathrm{Conv}_{\mathsf{H}}(X),\ \mu : (A,B,t) \mapsto (1-t)A \oplus tB$$

of producing a convex combination. Proposition 4.1 implies that the canonical representation δ is *affine* in the sense that

$$\delta\big((1-t)A \oplus tB\big) = (1-t)\delta(A) + t\delta(B)$$

for every $A, B \in \mathrm{Conv}_{\mathsf{H}}(X)$ and $t \in [0,1]$.

Propositions 2.1 and 4.1 will help us to establish the metric properties of the algebraic operations on $\mathrm{Conv}_{\mathsf{H}}(X)$.

Proposition 4.3. *Let $A, B, C, A', B' \in \mathrm{Conv}_{\mathsf{H}}(X)$ be five convex sets and $r \in \mathbb{R}$ and $t, t' \in [0,1]$ be three real numbers. Then*

1. $\mathsf{d}_{\mathsf{H}}(A \oplus B, A' \oplus B') \le \mathsf{d}_{\mathsf{H}}(A, A') + \mathsf{d}_{\mathsf{H}}(B, B')$*;*
2. $\mathsf{d}_{\mathsf{H}}(A \oplus B, A \oplus C) = \mathsf{d}_{\mathsf{H}}(B, C)$ *provided* $V_A^* \supset V_B^* \cup V_C^*$*;*
3. $\mathsf{d}_{\mathsf{H}}(\max\{A,B\}, \max\{A',B'\}) \le \max\{\mathsf{d}_{\mathsf{H}}(A,A'), \mathsf{d}_{\mathsf{H}}(B,B')\}$*;*
4. $\mathsf{d}_{\mathsf{H}}(r \cdot A, r \cdot B) = |r| \cdot \mathsf{d}_{\mathsf{H}}(A,B)$*;*
5. $\mathsf{d}_{\mathsf{H}}((1-t)A \oplus tB, (1-t')A \oplus t'B) = |t-t'|\mathsf{d}_{\mathsf{H}}(A,B)$*.*

Proof. All the items easily follow from Propositions 2.1, 4.1, and metric properties of algebraic operations on the hypermetric space $\bar{l}_\infty(\mathbb{S}^*)$. □

Observe that the metric components of the hypermetric space $\bar{l}_\infty(\mathbb{S}^*)$ are closed with respect to taking maximum and producing a convex combination. Moreover those operations are continuous on metric components of $\bar{l}_\infty(\mathbb{S}^*)$. With help of the canonical representation those properties of $\bar{l}_\infty(\mathbb{S}^*)$ transform into the corresponding properties of $\mathrm{Conv}_{\mathsf{H}}(X)$. In such a way we obtain

Corollary 4.4. *Each metric component $\mathcal{H}$ of $\mathrm{Conv}_{\mathsf{H}}(X)$ is closed under the operations of taking maximum and producing a convex combination. Moreover those operations are continuous on $\mathcal{H}$.*

Corollary 4.5. *Each metric component $\mathcal{H}$ of $\mathrm{Conv}_{\mathsf{H}}(X)$ is isometric to a convex max-subsemilattice of the Banach lattice $l_\infty(\mathbb{S}^*)$.*

A subset of a Banach lattice is called a *max-subsemilattice* is it is closed under the operation of taking maximum.

By a recent result of Banakh and Cauty [1], each non-locally compact closed convex subset of a Banach space is homeomorphic to an infinite-dimensional Hilbert space. This result combined with Corollary 4.5 implies:

Corollary 4.6. *Let X be a Banach space. A metric component $\mathcal{H}$ of $\mathrm{Conv}_{\mathsf{H}}(X)$ is homeomorphic to an infinite-dimensional Hilbert space if and only if $\mathcal{H}$ is not locally compact.*

This corollary reduces the problem of recognition of the topology of non-locally compact components of $\mathrm{Conv}_{\mathsf{H}}(X)$ to calculating their densities. This problem was considered in [3] and [4]). In particular, [3] contains the following characterization:

Proposition 4.7. *For a Banach space X and a metric component $\mathcal{H}$ of the space $\mathrm{Conv}_{\mathsf{H}}(X)$ the following conditions are equivalent:*

1. *$\mathcal{H}$ is separable;*
2. $\mathrm{dens}(\mathcal{H}) < \mathfrak{c}$*;*
3. *$\mathcal{H}$ contains a polyhedral convex set;*
4. *the characteristic cone $V_{\mathcal{H}}$ is polyhedral and belongs to $\mathcal{H}$;*

Remark 4.8. Each metric component of $\mathrm{Conv}_{\mathsf{H}}(X)$ being homeomorphic to a convex set, is connected and thus coincides with a connected component of $\mathrm{Conv}_{\mathsf{H}}(X)$. Hence there is no difference between metric and connected components of $\mathrm{Conv}_{\mathsf{H}}(X)$, so using the term *component* of $\mathrm{Conv}_{\mathsf{H}}(X)$ (without an adjective "metric" or "connected") will not lead to misunderstanding.

5. Operators between spaces of convex sets

Each linear continuous operator $T : X \to Y$ between normed spaces induces a map $\overline{T} : \mathrm{Conv}_{\mathsf{H}}(X) \to \mathrm{Conv}_{\mathsf{H}}(Y)$ assigning to each closed convex set $A \in \mathrm{Conv}_{\mathsf{H}}(X)$ the closure $\overline{T(A)}$ of its image $T(A)$ in Y. In this section we study properties of the induced operator $\overline{T}$. We start with algebraic properties that trivially follow from the linearity and continuity of the operator T.

Proposition 5.1. *If $T : X \to Y$ is a linear continuous operator between Banach spaces and $\overline{T} : \mathrm{Conv}_{\mathsf{H}}(X) \to \mathrm{Conv}_{\mathsf{H}}(X)$ is the induced operator, then*

(1) $\overline{T}(\max\{A,B\}) = \max\{\overline{T}(A), \overline{T}(B)\}$;
(2) $\overline{T}(r\cdot A) = r\cdot \overline{T}(A)$;
(3) $\overline{T}(A\oplus B) = \overline{T}(A) \oplus \overline{T(B)}$;
(4) $\overline{T}((1-t)A\oplus tB) = (1-t)\overline{T}(A) \oplus t\overline{T(B)}$;

for any sets $A, B \in \mathrm{Conv}_{\mathsf{H}}(X)$ and real numbers $r\in\mathbb{R}$ and $t\in[0,1]$.

We shall be mainly interested in the operators $\overline{T}$ induced by quotient operators T. We recall that for a closed linear subspace Z of a normed space X the quotient normed space $X/Z = \{x+Z : x\in X\}$ carries the quotient norm

$$\|x+Z\| = \inf_{y\in x+Z} \|y\|.$$

By $q : X\to X/Z$, $q : x\mapsto x+Z$ we shall denote the quotient operator and by $\bar{q} : \mathrm{Conv}_{\mathsf{H}}(X) \to \mathrm{Conv}_{\mathsf{H}}(X/Z)$ the induced operator between the spaces of closed convex sets.

For a closed convex set $C\subset X$ by C/Z we denote the image $q(C)\subset X/Z$. So, $\bar{q}(C) = \overline{C/Z}$. If $Z\subset V_C$, then the set C/Z is closed in X/Z and hence $\bar{q}(C) = C/Z$. Indeed, $Z\subset V_C$ implies that $C+Z=C$ and hence $C/Z = (X/Z)\setminus q(X\setminus C)$ is closed in X/Z being the complement of the set $q(X/\setminus C)$ which is open as the image of the open set $X\setminus C$ under the open map $q : X\to X/Z$.

We shall need the following simple reduction lemma:

Lemma 5.2. *Let Z be a closed linear subspace of a normed space X and A, B are non-empty closed convex subsets of X. If $Z\subset V_A\cap V_B$, then $\mathsf{d_H}(A,B) = \mathsf{d_H}(A/Z, B/Z)$.*

Proof. The inequality $\mathsf{d_H}(A/Z,B/Z) \le \mathsf{d_H}(A,B)$ follows from $\|q\|\le 1$. Assuming that $\mathsf{d_H}(A/Z,B/Z) < \mathsf{d_H}(A,B)$, we can find a point $a\in A$ with $\mathrm{dist}(a,B) > \mathsf{d_H}(A/Z,b/Z)$ or a point $b\in B$ with $\mathrm{dist}(b,A) > \mathsf{d_H}(A/Z,B/Z)$. Without loss of generality, we can assume that $\mathrm{dist}(a,B) > \mathsf{d_H}(A/Z,B/Z)$ for some point $a\in A$. Consider its image $a' = q(a)\in A/Z$ under the quotient operator $q : X\to X/Z$. By the definition of the Hausdorff metric, $\mathsf{d_H}(A/Z,B/Z) < \mathrm{dist}(a,B)$, there is a point $b'\in B/Z$ such that $\|b'-a'\| < \mathrm{dist}(a,B)$. By the definition of the quotient norm, there is a vector $z\in Z$ such that $q(z) = b'-a'$ and $\|z\| < \mathrm{dist}(a,B)$. Now consider the point $b = a+z$ and observe that $q(b) = q(a)+q(z) = a'+b'-a' = b'\in B/Z$ and hence $b\in q^{-1}(B/Z) = B+Z = B$. So, $\mathrm{dist}(a,B)\le\|a-b\| = \|z\| < \mathrm{dist}(a,B)$, which is a desired contradiction that completes the proof of the equality $\mathsf{d_H}(A,B) = \mathsf{d_H}(A/Z,B/Z)$. □

Corollary 5.3. *Let X be a normed space X, $\mathcal{H}$ be a component of the space $\mathrm{Conv}_{\mathsf{H}}(X)$, and Z be a closed linear subspace of X. If $Z\subset V_{\mathcal{H}}$, then the quotient operator*

$$\bar{q} : \mathcal{H}\to\mathcal{H}/Z, \ \ \bar{q} : C\mapsto C/Z,$$

maps isometrically the component $\mathcal{H}$ of $\mathrm{Conv}_{\mathsf{H}}(X)$ onto the component $\mathcal{H}/Z$ of $\mathrm{Conv}_{\mathsf{H}}(X/Z)$ containing some (equivalently, each) convex set C/Z with $C\in\mathcal{H}$.

6. Proof of Theorem 1

Let X be a Banach space and $\mathcal{H}$ be a component of the space $\mathrm{Conv}_{\mathsf{H}}(X)$.

If $\mathcal{H}$ contains no polyhedral convex set, then by Propositions 4.7, it has density $\mathrm{dens}(\mathcal{H})\ge\mathfrak{c}$. Consequently, $\mathcal{H}$ is not locally compact and by Corollary 4.6, $\mathcal{H}$ is homeomorphic to the non-separable Hilbert space $l_2(\kappa)$ of density $\kappa = \mathrm{dens}(\mathcal{H})\ge\mathfrak{c}$.

It remains to analyze the topological structure of $\mathcal{H}$ if it contains a polyhedral convex set. In this case Proposition 4.7 guarantees that the characteristic cone $V_{\mathcal{H}}$ belongs to $\mathcal{H}$ and is polyhedral in X. If $V_{\mathcal{H}} = X$, then $\mathcal{H} = \{X\}$ is a singleton. So, we assume that $V_{\mathcal{H}}\ne X$. Since the cone $V_{\mathcal{H}}$ is polyhedral, the the closed linear subspace $Z = -V_{\mathcal{H}}\cap V_{\mathcal{H}}$ has finite codimension in X. Then the quotient Banach space $\tilde{X} = X/Z$ is finite-dimensional. Let $q : X\to\tilde{X}$ be the quotient operator.

By Corollary 5.3, the component $\mathcal{H}$ is isometric to the component $\tilde{\mathcal{H}} = \mathcal{H}/Z$ of the space $\mathrm{Conv}_{\mathsf{H}}(\tilde{X})$ of closed convex subsets of the finite-dimensional Banach space $\tilde{X}$. The component $\tilde{\mathcal{H}}$ contains the polyhedral convex cone $V_{\tilde{\mathcal{H}}} = q(V_{\mathcal{H}})$ which has the property $-V_{\tilde{\mathcal{H}}}\cap V_{\tilde{\mathcal{H}}} = \{0\}$.

The cone $V_{\tilde{\mathcal{H}}}$ can be of two types.

1. The cone $V_{\tilde{\mathcal{H}}} = \{0\}$ is trivial. In this case $\mathcal{H}$ contains the closed linear subspace $Z = V_{\mathcal{H}}$ of finite codimension in X. Taking into account that $V_{\mathcal{H}}\ne X$, we conclude that $\dim(\tilde{X})\ge 1$. Depending on the value of $\dim(\tilde{X})$, we have two subcases.

1a. The dimension $\dim(\tilde{X}) = 1$ and hence $\mathcal{H}$ contains the linear subspace $Z = V_{\mathcal{H}}$ of codimension 1 in X. In this case $\tilde{\mathcal{H}}$ coincides with the space $\mathrm{BConv}_H(\tilde{X})$ of non-empty bounded closed convex subsets of the one-dimensional Banach space $\tilde{X}$ and hence $\tilde{\mathcal{H}}$ is isometric to the half-plane $\mathbb{R}\times\bar{\mathbb{R}}_+$.

1b. The dimension $\dim(\tilde{X}) \geq 2$ and hence $\mathcal{H}$ contains the linear subspace Z of codimension ≥ 2 in X. In this case $\tilde{\mathcal{H}}$ coincides with the space $\mathrm{BConv}_H(\tilde{X})$ of non-empty bounded closed convex subsets of the finite-dimensional Banach space $\tilde{X}$ of finite dimension $\dim(\tilde{X}) \geq 2$. By the result of Nadler, Quinn and Stavrakas [9], the space $\mathrm{BConv}_H(\tilde{X})$ is homeomorphic to the Hilbert cube manifold $Q \times \bar{\mathbb{R}}_+$.

2. The characteristic cone $V_{\tilde{\mathcal{H}}} \neq \{0\}$ is not trivial. This case has two subcases.

2a. $\dim(\tilde{X}) = \dim(V_{\tilde{\mathcal{H}}}) = 1$. In this case the component $\mathcal{H}$ (and its isometric copy $\mathcal{H}$) is isometric to the real line $\mathbb{R}$.

2b. $\dim(\tilde{X}) \geq 2$. In this case we shall prove that the component $\tilde{\mathcal{H}}$ (and its isometric copy $\mathcal{H}$) is homeomorphic to the separable Hilbert space l_2. This will follow from the separability of $\mathcal{H}$ and Corollary 4.6 as soon as we check that the space $\tilde{\mathcal{H}}$ is not locally compact. To prove this fact, it suffices for every positive $\varepsilon < 1$ to construct a sequence of closed convex sets $\{C_n\}_{n\in\mathbb{N}} \subset \tilde{\mathcal{H}}$ such that $\mathsf{d_H}(C_n, V_{\tilde{\mathcal{H}}}) \leq \varepsilon$ and $\inf_{n\neq m} \mathsf{d_H}(C_n, C_m) > 0$.

The cone $V_{\tilde{\mathcal{H}}}$ is polyhedral and hence is generated by for some finite set $E \subset \tilde{X} \setminus \{0\}$, see [8] or Theorem 1.1 of [17]. For every $e \in E$ the vector $-e$ does not belong to $V_{\tilde{\mathcal{H}}}$. Then the Hahn-Banach Theorem yields a linear functional $h_e \in X^*$ such that $h_e(-e) < \inf h_e(V_{\tilde{\mathcal{H}}}) = 0$. It can be shown that the functional $h = \sum_{e\in E} h_e$ has the property $h(v) > 0$ for all $v \in V_{\tilde{\mathcal{H}}} \setminus \{0\}$.

Since $\dim(\tilde{X}) \geq 2$ and $V_{\tilde{\mathcal{H}}} \neq \tilde{X}$, we can find a non-zero linear continuous functional $f : \tilde{X} \to \mathbb{R}$ such that $\sup f(V_{\tilde{\mathcal{H}}}) = 0$ and the intersection $f^{-1}(0) \cap V_{\tilde{\mathcal{H}}}$ contains a non-zero vector $x \in \tilde{X}$. Multiplying x by a suitable positive constant, we can assume that $h(x) = 1$. Since $h^{-1}(0) \cap V_{\tilde{\mathcal{H}}} = \{0\} \neq f^{-1}(0) \cap V_{\tilde{\mathcal{H}}}$, the functionals h and f are distinct and hence there is a vector $y \in h^{-1}(0) \setminus f^{-1}(0)$ with norm $\|y\| = \varepsilon$. Replacing y by $-y$, if necessary, we can assume that $f(y) > 0$.

For every $n \in \omega$ consider the point $c_n = 3^n x + y$ and the closed convex set

$$C_n = \max\big\{V_{\tilde{\mathcal{H}}}, \{c_n\}\big\} = \overline{\mathrm{conv}}(V_{\tilde{\mathcal{H}}} \cup \{c_n\}) \subset \tilde{X}.$$

It follows from $x \in V_{\tilde{\mathcal{H}}}$ and $\mathrm{dist}(c_n, V_{\tilde{\mathcal{H}}}) \leq \mathrm{dist}(3^n x + y, 3^n x) = \|y\| = \varepsilon$ that $\mathsf{d_H}(C_n, V_{\tilde{\mathcal{H}}}) \leq \varepsilon$.

We claim that $\inf\limits_{n\neq m} \mathsf{d_H}(C_n, C_m) \geq \delta$ where

$$\delta = \frac{1}{2} f(y) \leq \frac{1}{2}\|y\| = \frac{1}{2}\varepsilon < \frac{1}{2}.$$

This will follow as soon as we check that $\mathrm{dist}(c_n, C_m) \geq \delta$ for any numbers $n < m$.

Assuming conversely that $\mathrm{dist}(c_n, C_m) < \delta$ and taking into account that the convex set $\mathrm{conv}(V_{\tilde{\mathcal{H}}} \cup \{c_m\})$ is dense in C_m, we can find a point $c \in \mathrm{conv}(V_{\tilde{\mathcal{H}}} \cup \{c_m\})$ such that $\mathrm{dist}(c_n, c) < \delta$. The point c belongs to the convex hull of the set $V_{\tilde{\mathcal{H}}} \cup \{c_m\}$ and hence can be written as a convex combination $c = tc_m + (1-t)v = t(3^m x + y) + (1-t)v$ for some $t \in [0, 1]$ and $v \in V_{\tilde{\mathcal{H}}}$. Observe that $h(c_n) = h(3^n x + y) = 3^n h(x) + h(y) = 3^m \cdot 1 + 0 = 3^n$ while $h(c) = th(c_m) + (1-t)h(v) \geq th(c_m) = 3^m t$. Then

$$3^m t - 3^n \leq h(c) - h(c_n) \leq |h(c) - h(c_n)| \leq \|h\| \cdot \|c - c_n\| < 1 \cdot \delta$$

and hence

$$t < 3^{n-m} + 3^{-m}\delta \leq \frac{1}{3} + \frac{1}{3}\delta < \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

Next, we apply the functional f to the points c_n and c. Since $f(x) = 0$, we get $f(c_n) = f(3^n x + y) = f(y) = 2\delta$. On the other hand, $f(V_{\tilde{\mathcal{H}}}) \subset (-\infty, 0]$ implies $f(v) \leq 0$ and hence

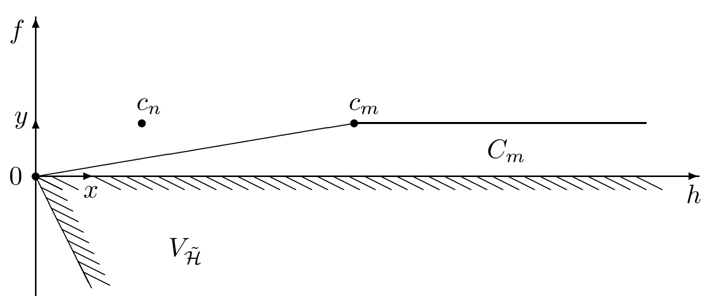
$$f(c) = f(tc_m + (1-t)v) = tf(3^m x + y) + (1-t)f(v) = tf(y) + (1-t)f(v) \leq tf(y) = 2\delta t.$$

Then

$$\delta = 2\delta(1 - \tfrac{1}{2}) < 2\delta(1-t) \leq |f(c_n) - f(c)| \leq \|f\| \cdot \|c_n - c\| < \delta,$$

which is a desired contradiction.

The above proof can be visualized in the following picture:



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